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Computer Science

T07 – Week 8

Dynamic Programming

CS3230 – Design and Analysis of Algorithms

Recap from T04

Equality Testing Problem

The key is to break down into the various cases (this I think everyone can do):

- $A = B$ and decided $A = B$: 100% correct
- $A = B$ and decided $A! = B$: 100% correct
- $A! = B$ and decided $A = B$: $1 - 1/n$ correct
- $A! = B$ and decided $A! = B$: 100% correct

The trick to break it down into the various cases.

Random Partition of a Graph

Similarly, it is to break down into the various cases. After getting the form, then try to break the math.

Admin

1. Midterm Exam @ MPSH 1B (80mins): 13-Mar-2025, 14:00-16:00; Arrive at venue by 14:00, Exam starts ~14:10
   - Info - Up until Randomized Algorithms; No calculators.
   - Seat Map
   - Seat Plan
2. Anything mentioned in (lectures, tutorials, assignments) would be ok to be quoted; Everything else should be proved before using.
   - Take note of this as some of you used some beyond the scope of our class (such as Akra–Bazzi, for which that is not accepted without proof.).
3. All the best for your midterms!

Key Ideas in Dynamic Programming (DP)

- **Optimal substructure**: Solve recursively by breaking into subproblems.
- **Few unique subproblems**: Avoid redundant recomputation.

Two Approaches:

- **Top-down (Memoization)**: Store computed results to reuse in $O(1)$.
- **Bottom-up (Tabulation)**: Solve iteratively from base cases.

Both methods improve efficiency by **avoiding redundant** work.

Convex Polygon Triangulation

Minimize the total weight of $n-2$ triangles in the optimal triangulation, considering:

- Given a convex polygon with $n \geq 2$ vertices labeled $1, 2, \ldots, n$
- Divide the polygon into $n-2$ triangles.
- A triangle (x, y, z) has weight $W(x, y, z)$ (an $O(1)$ black-box function).
- Multiple triangulations exist.

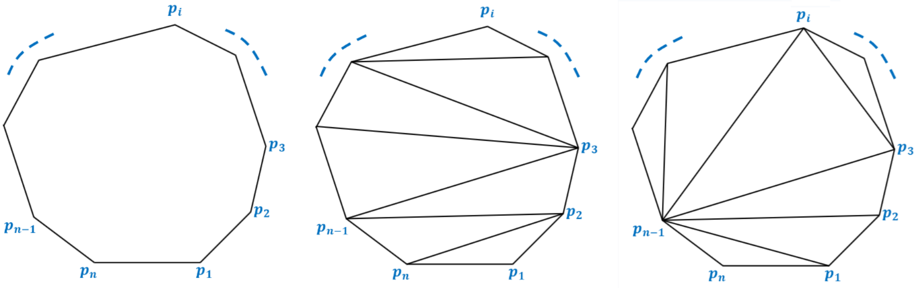

Figure 1: Two triangulation examples (middle, right).

Question 1 [G]/[P1]

Let $TRI(x, y)$ be a function to triangulate a polygon with minimum weight sum, but we only consider the vertices in the range of $(x, x+1, x+2, \dots, y)$. So our problem can be solved by calling $TRI(1, n)$. Your first task is to write a recursive formula of $TRI(x, y)$.

a. Find the base case of $TRI(x, y)$

b. Find the recursive case of $TRI(x, y)$

Hint: It calls $TRI(x', y')$ where $x < x'$ or $y' < y$.

Answer

$$TRI(x,y) = \begin{cases} 0, & \text{if } y - x = 1 \\ \min\limits_{k \in [x+1, y-1]} \left[TRI(x,k) + W(x,k,y) + TRI(k,y)\right]. & \text{otherwise} \end{cases}$$

a. **Base Case**: Cannot triangulate a line (adjacent vertices x and y).

b. **Recursive Case**: Try all triangulations in any order in the recurrence:
   - Subproblems $TRI(x,k)$ and $TRI(k,y)$
   - Triangle (x,k,y) with weight $W(x,k,y)$

Illustration

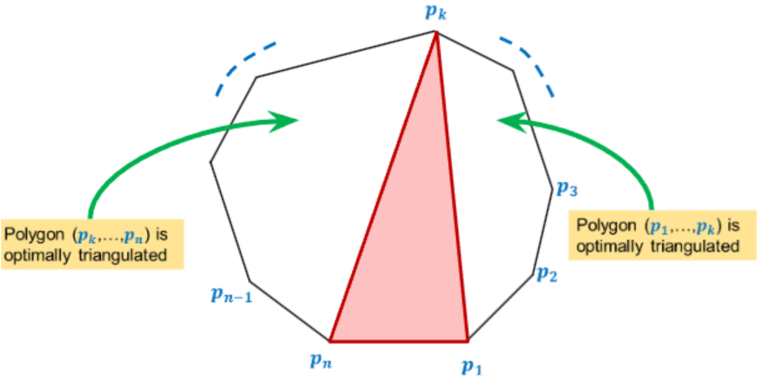

Figure 2: Optimal substructure

Question 2 [G]

What is the time complexity of this recursive formula $TRI(1, n)$, if implemented verbatim.

a. $O(n^2)$

b. $O(n^3)$

c. $O(3^n)$

Answer

Let $T(n)$ be the worst-case running time of $TRI(1, n)$.

$$T(2) = c, \quad \text{when } y - x = 1.$$

Expanding the recurrence for $T(n), T(n-1)$:

$$\begin{aligned} T(n) &= (T(2) + T(n-1) + c) + (T(3) + T(n-2) + c) \\ &\quad + \dots + (T(n-2) + T(3) + c) + (T(n-1) + T(2) + c) \\ T(n-1) &= (T(2) + T(n-2) + c) + (T(3) + T(n-3) + c) \\ &\quad + \dots + (T(n-2) + T(2) + c) \end{aligned}$$

Subtracting $T(n-1)$ from $T(n)$:

$$\begin{aligned} T(n) - T(n-1) &= 2T(n-1) + c \\ &\implies T(n) = 3T(n-1) + c \\ &\implies T(n) \approx 3^n \in O(3^n). \end{aligned}$$

Question 3 [P2]

Which one is the correct explanation regarding the findings from (Q2)?

a. It has 3^n non-overlapping subproblems, and each call runs in $\Theta(1)$.

b. It has n^2 non-overlapping subproblems, and each call runs in $\Theta\left(\frac{3^n}{n^2}\right)$.

c. It has n^2 subproblems, but there are many overlaps.

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Answer

It has n^2 subproblems with significant overlap, making a Dynamic Programming solution necessary for efficiency.

Question 4 [G]

Design a Dynamic Programming (DP) solution for **Convex Polygon Triangulation** problem.

a. Using Top-Down DP

b. Using Bottom-Up DP

Answer

Using Top-Down DP

Use a 2D $memo$ table of size $n \times n$ ($O(n^2)$ space).

Algorithm

1. $TRI(x, y)$ is previously computed: return $memo[x][y]$
2. otherwise recursively solve $O(n)$ subproblems:
   a. Compute the $\min$ for $x < k < y : TRI(x, k) + W(x, k, y) + TRI(k, y)$
   b. Store it in $memo[x][y]$

Analysis

- $O(n^2)$ different subproblems
- each sub-problem is only computed once in $O(n)$
- so the total time complexity is $O(n^2 \times n) = O(n^3)$.

Illustration's Weights

Different weight functions can be used. Standard implementations typically define a triangle's weight as its perimeter, the sum of its side lengths. For illustration, we randomly assign weights to each triangle.

Table 1: Truncated table of randomly generated weights for animations and illustrations.

x	k	y	$W(x,k,y)$
0	1	2	3
0	1	3	1
0	1	4	7
0	1	5	4
0	2	3	2

Illustration

Show animation of the memoization table: T07.q4a.gif.

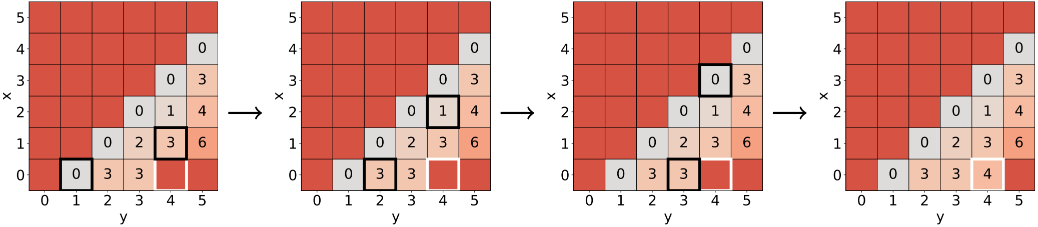

Figure 3: $memo$ table: $TRI(0,4)$ with its subproblems in row $TRI(0,?)$ and column $TRI(?,4)$.

Using Bottom-Up DP

Use a 2D TRI DP table of size $n \times n$ ($O(n^2)$ space, same as $memo$), but now we must determine the **correct filling order** (**topological order** of the underlying recursion DAG).

Algorithm

1. Base Case: For each $x \in [1..n-1]$, set $TRI[x][x+1] = 0$. This is **one index away** from the anti-diagonal of the $n \times n$ DP table.
2. Recursive Case: Fill the table anti-diagonally, starting from **2 indices away** from the anti-diagonal. Each $TRI(x, y)$ needs to compute the min over previously computed values in its row and column, requiring a anti-diagonal filling order.

Analysis

Overall time complexity is $O(n^3)$,

- which is the same as Top-Down DP approach,
- Bottom-Up method can benefit from reduced recursion overhead.

Implementation

```python
def compute_bottomup(n, w):
    TRI = [[ -1 ] * n for _ in range(n)]  # Initialize the DP table
    for x in range(n - 1):  # Base case, notice the 0-based indexing
        TRI[x][x + 1] = 0

    # Fill the table anti-diagonally
    for delta in range(2, n):  # Delta is the gap between x and y
        for x in range(n - delta):  # Iterate over all valid x
            y = x + delta
            t = float('inf')
            for k in range(x + 1, y):  # min over all x < k < y
                t = min(t, TRI[x][k] + w(x, k, y) + TRI[k][y])
            TRI[x][y] = t

    return TRI[0][n - 1]
```

Illustration

Show animation of the DP table: T07.q4b.gif.

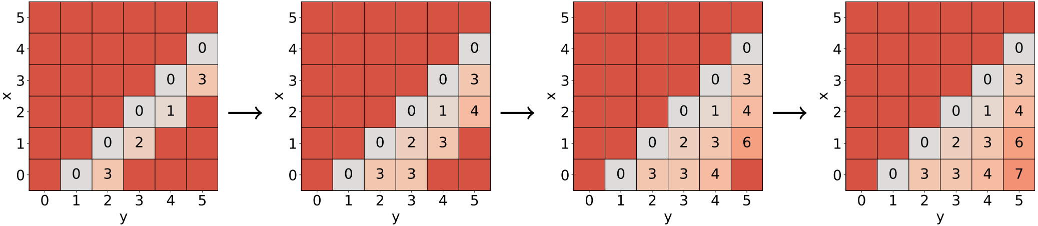

Figure 4: TRI DP table: Progressing through each anti-diagonal.

Practical [Optional]

Practical repo: To help you further your understanding, not compulsory; Work for Snack!

1. Implement `compute_topdown`.
2. Check that you get this output:

```
Top-down == Bottom-up: 7
```