



**NUS**  
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| **Computing**

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*Computer Science*

T01 – Week 2

# Introduction and Asymptotic Analysis

*CS3230 TG19*

- › **[Pioneer JC 2009-2010]** Took 'A' levels and fell in love with Computing
  - ›› H2 Computing, Interested in research and in AI.
- › **[B.Com. NUS 2013-2018]** Not so long ago I was in your seat
  - ›› A\*STAR Scholarship, Turing Programme
  - ›› [University of Southern California, 2016] Student Exchange
- › **[PhD. NUS 2018-2023]** PhD in AI/ML
  - ›› My research is in AI/Machine Learning regarding optimization, scaling and robustness.
- › **[Lecturer]** Applied for jobs and got 2 - Industry and another in NUS SoC
  - ›› If you are interested to do a FYP, feel free to chat with me!
  - ›› You are welcome to check my profile & research: <https://eric-han.com>
  - ›› Some of the courses I taught: CS2109S(3), CS3244(1), CS1010(1), CS3217(1), CS3243(2), CS3203(5), CS2030(1)
  - ›› Teaching this semester: CS3230(1), CS2103/T(1)

About CS3230: <insert personal exp. here>; Goal: Learn algorithms, do well.

## Expectations of you

- 1 Fill seats from the front.
- 2 Good students are always prepared:
  - a. Attempt your Tutorial
  - b. Review lecture content
  - c. Be on time
- 3 Refrain from taking pictures of the slides.
  - a. Learn to take good notes.
  - b. Slides/notes will be distributed; Created the main deck, with customizations...

## Commitment from me

- 1 Be available for your learning as much as possible.
- 2 Strive to make the lessons interesting and fun.
- 3 Pass on a good foundation in Algorithms (not just the **A+**).

Any comments or suggestions for the lessons welcome!

- › **Grading** (20 marks, 5% of final grade):
  - ›› **Attendance** (12 marks): *1 mark* per session (12 total).
  - ›› **Participation** (10 marks): *5 marks* for each of **two presentations**. Identified by [P1/2/3].
- › **Tutorial Discussion**: Learning is **social** and I hope that we are able to build friendships in this class. Identified by [G].
- › **Policies**: [Tutorials & Assignment Policy](#): Plagiarism (no AI tools).
- › **Assignments**: Graded by me.
- › **Consultations**: [Wed 10-11 AM](#).
- › **Telegram Group**: [Join for updates](#).
- › **Welcome Survey**: [Get to know you better](#).
- › **Questions?**: Use ChatGPT, Telegram Group / PM `@Eric_Vader`.



Figure 1: Telegram Group



Figure 2: Welcome Survey



Figure 3: PPT Schedule

Asymptotic Analysis  $\Omega, \Theta, O$ 

We say	if $\exists c, c_1, c_2, n_0 > 0$ s.t. $\forall n \geq n_0$	In other words
$f(n) \in O(g(n))$	$0 \leq f(n) \leq c \cdot g(n)$	$g$ is an <b>upper</b> bound on $f$
$f(n) \in \Omega(g(n))$	$0 \leq c \cdot g(n) \leq f(n)$	$g$ is a <b>lower</b> bound on $f$
$f(n) \in \Theta(g(n))$	$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$	$g$ is a <b>tight</b> bound on $f$

Asymptotic Analysis  $o, \omega$ 

We say	if $\forall c > 0, \exists n_0 > 0$ s.t. $\forall n \geq n_0$	In other words
$f(n) \in o(g(n))$	$0 \leq f(n) < c \cdot g(n)$	$g$ is a <b>strict upper</b> bound on $f$
$f(n) \in \omega(g(n))$	$0 \leq c \cdot g(n) < f(n)$	$g$ is a <b>strict lower</b> bound on $f$

Assume  $f(n), g(n) > 0$ , show:

a.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) \in o(g(n))$  — this has already been shown in lec01b.

b.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in O(g(n))$  [P1]

c.  $0 < \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in \Theta(g(n))$

d.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0 \Rightarrow f(n) \in \Omega(g(n))$

e.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) \in \omega(g(n))$

### Recap

- › Definition of Limit for functions

Assume  $f(n), g(n) > 0$ , show:

a.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) \in o(g(n))$  — this has already been shown in lec01b.

b.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in O(g(n))$  [P1]

c.  $0 < \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in \Theta(g(n))$

d.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0 \Rightarrow f(n) \in \Omega(g(n))$

e.  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) \in \omega(g(n))$

### Recap

- › Definition of Limit for functions

$$\lim_{n \rightarrow \infty} f(n) = L \iff (\forall \epsilon > 0)(\exists n_0 > 0)(n > n_0 \implies |f(n) - L| < \epsilon).$$

- › How to proof?



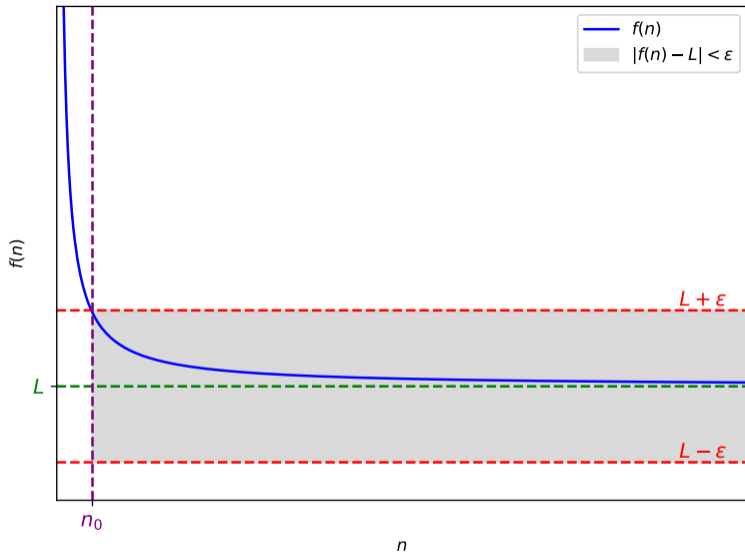


Figure 4:  $\lim_{n \rightarrow \infty} f(n) = L \iff (\forall \epsilon > 0)(\exists n_0 > 0)(n > n_0 \implies |f(n) - L| < \epsilon)$ .

## Answer 1b

Proof  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in O(g(n))$ .

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = z \implies (\forall \epsilon > 0)(\exists n_0 > 0) \left| \frac{f(n)}{g(n)} - z \right| < \epsilon, \quad (\text{By definition})$$

$$\implies -\epsilon < \frac{f(n)}{g(n)} - z < \epsilon$$

$$\implies z - \epsilon < \frac{f(n)}{g(n)} < z + \epsilon$$

$$\implies \frac{f(n)}{g(n)} \leq c \quad (\text{Let } c = z + \epsilon + \dots)$$

$$\implies (\forall \epsilon > 0)(\exists n_0 > 0) f(n) \leq c \cdot g(n) \quad (g(n) > 0)$$

$$\implies (\exists n_0 > 0) f(n) \leq c \cdot g(n) \quad (\epsilon > 0 \text{ arbitrary})$$

$$\implies f(n) \in O(g(n)). \quad (\text{By definition of } O \text{ notation})$$

**Answer 1c**

Proof  $0 < \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \in \Theta(g(n))$ .

$$0 < \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \in O(g(n)) \quad (\text{Upper bound})$$

$$\text{and } f(n) \in \Omega(g(n)). \quad (\text{Lower bound})$$

$$\implies f(n) \in \Theta(g(n)). \quad (\text{By definition of } \Theta \text{ notation})$$

**Answer 1d**

Proof  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0 \implies f(n) \in \Omega(g(n))$ .

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = z \implies \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \frac{1}{z}, \quad (\text{Flip } f(n) \text{ and } g(n))$$

$$\implies g(n) \in O(f(n)), \quad (\text{By the explanation above})$$

$$\implies f(n) \in \Omega(g(n)). \quad (\text{By complementarity property})$$

**Answer 1e**

Proof  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty \implies f(n) \in \omega(g(n))$ .

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty \implies \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 0, \quad (\text{Flip } f(n) \text{ and } g(n))$$

$$\implies g(n) \in o(f(n)), \quad (\text{By the definition of } o \text{ notation})$$

$$\implies f(n) \in \omega(g(n)). \quad (\text{By complementarity property})$$

**Moral of the story**

- › Be lazy, proof a single case and use properties.

Assume  $f(n), g(n) > 0$ , show:

**a.** Reflexivity

»  $f(n) \in O(f(n))$

»  $f(n) \in \Omega(f(n))$

»  $f(n) \in \Theta(f(n))$

**b.** Transitivity [P2]

»  $f(n) \in O(g(n))$  and  $g(n) \in O(h(n))$  implies  $f(n) \in O(h(n))$

» Do the same for  $\Omega$ ,  $\Theta$ ,  $o$ ,  $\omega$

**c.** Symmetry

»  $f(n) \in \Theta(g(n))$  iff  $g(n) \in \Theta(f(n))$

**d.** Complementarity

»  $f(n) \in O(g(n))$  iff  $g(n) \in \Omega(f(n))$

»  $f(n) \in o(g(n))$  iff  $g(n) \in \omega(f(n))$

## Answer 2a

### Reflexivity

- ›  $f(n) \in O(f(n))$ 
  - ›› Taking  $c = 1$  (any constant  $\geq 1$ ),  $n_0 = 1$ , we have  $\forall n \geq n_0$ ,
  - ››  $f(n) \leq (1 \cdot f(n) = c \cdot f(n))$ .
- ›  $f(n) \in \Omega(f(n))$ 
  - ›› Taking  $c = 1$  (any positive constant  $\leq 1$ ),  $n_0 = 1$ , we have  $\forall n \geq n_0$ ,
  - ››  $(c \cdot f(n) = 1 \cdot f(n)) \leq f(n)$ .
- ›  $f(n) \in \Theta(f(n))$ 
  - ›› Taking  $c_1 = 1$  (any positive constant  $< 1$ ),  $c_2 = 1$ ,  $n_0 = 1$ , we have  $\forall n \geq n_0$ ,
  - ››  $(c_1 \cdot f(n) = 1 \cdot f(n)) \leq f(n) \leq (1 \cdot f(n) = c_2 \cdot f(n))$ .

## Answer 2b

Proof  $f(n) \in O(g(n)), g(n) \in O(h(n)) \implies f(n) \in O(h(n))$

$$f(n) \in O(g(n)) \implies (\exists c_{fg} > 0, n_{0fg} > 0)(\forall n \geq n_{0fg}), \quad (1)$$
$$f(n) \leq c_{fg} \cdot g(n).$$

$$g(n) \in O(h(n)) \implies (\exists c_{gh} > 0, n_{0gh} > 0)(\forall n \geq n_{0gh}), \quad (2)$$
$$g(n) \leq c_{gh} \cdot h(n).$$

$$\begin{aligned} \text{From (1), (2)} &\implies (\forall n \geq \max(n_{0fg}, n_{0gh})), \\ &f(n) \leq c_{fg} \cdot g(n) \leq c_{fg} \cdot c_{gh} \cdot h(n) \\ &\implies f(n) \leq c_{fg} \cdot g(n) \leq c \cdot h(n) && \text{(Let } c = c_{fg} \cdot c_{gh}\text{)} \\ &\implies (\forall n \geq n_0) && \text{(Let } n_0 = \max(n_{0fg}, n_{0gh})\text{)} \\ &f(n) \leq c_{fg} \cdot g(n) \leq c \cdot h(n) \\ &\implies f(n) \leq c \cdot h(n) \\ &\implies f(n) \in O(h(n)). \end{aligned}$$

**Do the same for  $\Omega$ ,  $\Theta$ ,  $o$ ,  $\omega$**

- › Same, just change  $\leq$  to  $\geq$ ,  $=$ ,  $<$ ,  $>$ , respectively. Here,  $=$  for  $\Theta$  denotes that we need to do both  $\geq$  and  $\leq$  bounds.



## Answer 2c

Prove  $f(n) \in \Theta(g(n)) \iff g(n) \in \Theta(f(n))$

$$f(n) \in \Theta(g(n)) \implies (\exists c_1, c_2 > 0, n_0 > 0)(\forall n \geq n_0), \\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n).$$

$$\text{From } f(n) \leq c_2 \cdot g(n) \implies \frac{1}{c_2} \cdot f(n) \leq g(n) \quad (\text{Divide by } c_2)$$

$$\implies c'_1 \cdot f(n) \leq g(n) \quad (\text{Let } c'_1 = \frac{1}{c_2}), (1)$$

$$\text{From } c_1 \cdot g(n) \leq f(n) \implies g(n) \leq \frac{1}{c_1} \cdot f(n). \quad (\text{Divide by } c_1)$$

$$\implies g(n) \leq c'_2 \cdot f(n) \quad (\text{Let } c'_2 = \frac{1}{c_1}), (2)$$

$$\text{From (1), (2)} \implies c'_1 \cdot f(n) \leq g(n) \leq c'_2 \cdot f(n) \\ \implies g(n) \in \Theta(f(n)).$$

## Answer 2d

Proof  $f(n) \in O(g(n)) \iff g(n) \in \Omega(f(n))$

**Forward Direction: Suppose  $f(n) \in O(g(n))$**

$$\begin{aligned} f(n) \in O(g(n)) &\implies (\exists c > 0, n_0 > 0)(\forall n \geq n_0), \\ & f(n) \leq c \cdot g(n) \\ &\implies \frac{1}{c} \cdot f(n) \leq g(n) && \text{(Divide by } c) \\ &\implies c' \cdot f(n) \leq g(n) \implies g(n) \in \Omega(f(n)). && \text{(Let } c' = \frac{1}{c}) \end{aligned}$$

**Reverse Direction: Suppose  $g(n) \in \Omega(f(n))$**

$$\begin{aligned} g(n) \in \Omega(f(n)) &\implies (\exists c' > 0, n_0 > 0)(\forall n \geq n_0), \\ & c' \cdot f(n) \leq g(n) \\ &\implies f(n) \leq c \cdot g(n) \implies f(n) \in O(g(n)). && \text{(Let } c = \frac{1}{c'}) \end{aligned}$$

Proof  $f(n) \in o(g(n)) \iff g(n) \in \omega(f(n))$

- › Same as above, just change  $\leq$  to  $<$ .

Which of the following statement(s) is/are True?

- a.  $3^{n+1} \in O(3^n)$
- b.  $4^n \in O(2^n)$
- c.  $2^{\lfloor \log n \rfloor} \in \Theta(n)$  (we assume  $\log$  is in base 2)
- d. For constants  $i, a > 0$ , we have  $(n + a)^i \in O(n^i)$

### Recap

How to go about showing True/False?

**Answer 3a**

$3^{n+1} \in O(3^n)$ : True.

- › Taking  $c = 3, n_0 = 1$ , we have  $\forall n \geq n_0$ ,
- ›  $3^{n+1} \leq 3 \cdot 3^n = c \cdot 3^n$ .
- ›  $3^{n+1} \in O(3^n)$ .

**Answer 3b**

$4^n \in O(2^n)$ : False.

- › For all  $c \geq 1, n_0 = c$ , we have  $\forall n \geq n_0$ ,
- ›  $(4^n = (2^2)^n = (2^n)^2 = 2^n \cdot 2^n) \geq c \cdot 2^n$ , i.e., we cannot upper bound  $4^n$  with constant times  $2^n$ .

### Answer 3b (Limit)

Another approach is to use limits, using  $\lim_{n \rightarrow \infty} \frac{4^n}{2^n} < \infty \implies 4^n \in O(2^n)$ :

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{4^n}{2^n} &= \lim_{n \rightarrow \infty} \frac{(2^2)^n}{2^n} \\ &= \lim_{n \rightarrow \infty} \frac{2^{2n}}{2^n} \\ &= \lim_{n \rightarrow \infty} 2^{2n-n} \\ &= \lim_{n \rightarrow \infty} 2^n.\end{aligned}$$

Since  $2^n \rightarrow \infty$  as  $n \rightarrow \infty$ , the limit diverges. Therefore,  $\frac{4^n}{2^n} \notin O(2^n)$ .

**Answer 3c**

$2^{\lceil \log n \rceil} \in \Theta(n)$  (we assume  $\log$  is in base 2): True.

- › Taking  $c_1 = \frac{1}{2}$ ,  $c_2 = 1$ ,  $n_0 = 1$  ( $\log 0$  is undefined), we have  $\forall n \geq n_0$ ,
- ›  $(c_1 \cdot n = \frac{1}{2} \cdot n) \leq 2^{\lceil \log n \rceil} \leq (1 \cdot n = c_2 \cdot n)$ .
- ›  $2^{\lceil \log n \rceil} \in \Theta(n)$ .

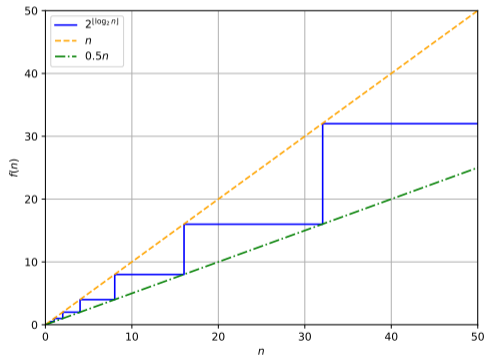


Figure 5: Illustration of Q3c

### Answer 3d

For constants  $i, a > 0$ , we have  $(n + a)^i \in O(n^i)$ : True.

- › Taking  $c = 2^i, n_0 = a$ , we have  $\forall n \geq n_0$ ,
- ›  $(n + a)^i \leq (n + n)^i = (2n)^i = 2^i \cdot n^i = c \cdot n^i$ .
- ›  $(n + a)^i \in O(n^i)$ .

### Answer 3d (Limit)

Another approach is to use limits, using  $\lim_{n \rightarrow \infty} \frac{(n+a)^i}{n^i} < \infty \implies (n + a)^i \in O(n^i)$ :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n + a)^i}{n^i} &= \lim_{n \rightarrow \infty} \left( \frac{n + a}{n} \right)^i && \text{(Simplify the ratio)} \\ &= \lim_{n \rightarrow \infty} \left( 1 + \frac{a}{n} \right)^i && \text{(Simplify)} \\ &= \left( \lim_{n \rightarrow \infty} \left( 1 + \frac{a}{n} \right) \right)^i && \text{(Limits Power Law)} \\ &= 1^i \implies 1 \implies \Theta(n^i) \implies O(n^i) && \text{(As } \lim_{n \rightarrow \infty} \frac{a}{n} = 0 \text{)} \end{aligned}$$



Which of the following statement(s) is/are True?

$$2^{\log_2 n} \in$$

- a.  $O(n)$
- b.  $\Omega(n)$
- c.  $\Theta(\sqrt{n})$
- d.  $\omega(n)$

### Recap

- › Inverse Property of Logarithms

Which of the following statement(s) is/are True?

$$2^{\log_2 n} \in$$

- a.  $O(n)$
- b.  $\Omega(n)$
- c.  $\Theta(\sqrt{n})$
- d.  $\omega(n)$

### Recap

› Inverse Property of Logarithms

$$a^{\log_a b} = b$$

Which of the following statement(s) is/are True?

$$2^{\log_2 n} \in$$

- a.  $O(n)$
- b.  $\Omega(n)$
- c.  $\Theta(\sqrt{n})$
- d.  $\omega(n)$

### Recap

› Inverse Property of Logarithms

$$a^{\log_a b} = b$$

### Answer

$2^{\log_2 n} = n \in O(n)$ , and also  $n \in \Omega(n)$ , but  $n \notin \omega(n)$  (why? - if you can answer, you understand  $\omega$  vs  $\Omega$ ).

How about  $2^{\log_4 n} \in ?$

How about  $2^{\log_4 n} \in ?$

### Answer

First, rewrite the logarithm from one base to another base:

$$\triangleright \log_4 n = \frac{\log_2 n}{\log_2 4} = \frac{\log_2 n}{2},$$

Thus,  $2^{\log_4 n} = 2^{\frac{\log_2 n}{2}} = (2^{\log_2 n})^{\frac{1}{2}} = n^{\frac{1}{2}} = \sqrt{n}$ .

**a.**  $O(n)$  True.

$\gg 2^{\log_4 n} = \sqrt{n} \in O(n)$ , taking  $c = 1, n_0 = 1$ .

**b.**  $\Omega(n)$  False.

**c.**  $\Theta(\sqrt{n})$  True.

$\gg 2^{\log_4 n} = \sqrt{n} \in \Theta(\sqrt{n})$ , taking  $c_1 = 1$  (or smaller),  $c_2 = 1$  (or larger),  $n_0 = 1$ .

**d.**  $\omega(n)$  False.

Rank the following functions by their order of growth. (But if any two (or more) functions have the same order of growth, group them together).

1  $f_1(n) = \log n$

2  $f_2(n) = n!$

3  $f_3(n) = 2^n + n$

4  $f_4(n) = n^{2.3} + 16n + f_1(n)$

5  $f_5(n) = \log(n^2)$

6  $f_6(n) = \ln(n^{2n})$

**Answer**

$$1 \quad f_1(n) = \log n$$

$$2 \quad f_2(n) = n!$$

$$3 \quad f_3(n) = 2^n + n$$

$$\gg f_3(n) \in \Theta(2^n)$$

$$4 \quad f_4(n) = n^{2.3} + 16n + f_1(n)$$

$$\gg f_4(n) \in \Theta(n^{2.3}).$$

$$5 \quad f_5(n) = \log(n^2)$$

$$\gg f_5(n) = \log(n^2) = 2 \log n, \text{ hence the same order of growth as } f_1(n).$$

$$6 \quad f_6(n) = \ln(n^{2n})$$

$$\gg f_6(n) = 2n \ln(n) \in \Theta(n \ln n).$$

**Answer**

$$1 \quad f_1(n) = \log n$$

$$2 \quad f_2(n) = n!$$

$$3 \quad f_3(n) = 2^n + n$$

$$\gg f_3(n) \in \Theta(2^n)$$

$$4 \quad f_4(n) = n^{2.3} + 16n + f_1(n)$$

$$\gg f_4(n) \in \Theta(n^{2.3}).$$

$$5 \quad f_5(n) = \log(n^2)$$

$$\gg f_5(n) = \log(n^2) = 2 \log n, \text{ hence the same order of growth as } f_1(n).$$

$$6 \quad f_6(n) = \ln(n^{2n})$$

$$\gg f_6(n) = 2n \ln(n) \in \Theta(n \ln n).$$

Ordering  $f_2, f_3$ :

$$\triangleright f_2(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1.$$

$$\triangleright \text{simplified } f_3(n) = 2^n = 2 \cdot 2 \cdot 2 \cdot \dots \cdot 2.$$

$$\triangleright \text{can show by induction that for } n \geq 4, n! \geq \frac{n}{4} \cdot 2^n.$$

Therefore, with respect to order of growth, we have:

$$(f_1(n) = f_5(n)) \leq f_6(n) \leq f_4(n) \leq f_3(n) \leq f_2(n)$$



Practical repo: To help you further your understanding, not compulsory; Work for Snack!

- 1 Visualize the growth of various functions in Q5 and
- 2 Plot the threshold ( $n = ?$ ) for which the asymptotic relations can be observed.

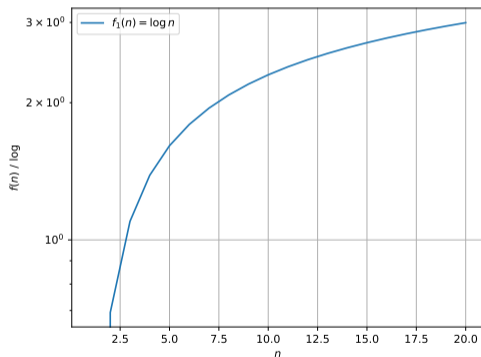


Figure 6: Functions compared - visualization to aid understanding/not proof.